ROTATION MATRIX

Example 1. *The matrix representing the linear transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$*, where* T *is the rotation in the counter-clockwise direction by degree* θ *in* $\mathbb{R}^2$*, is given by*

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Proof. Recall the Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This is the same as the vector $(x, y) = (\cos\theta, \sin\theta)$ in the $x - y$ plane. This is a vector of unit length because $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$. The angle between (x, y) and the x-axis is θ because $y/x = \tan\theta$.

Now take an arbitrary vector (a, b) in the $x - y$ plane. This is the same as the complex number

$$a + ib = \sqrt{a^2 + b^2}[(a/\sqrt{a^2 + b^2}) + i(b/\sqrt{a^2 + b^2})] = Re^{i\beta},$$

where $R = \sqrt{a^2 + b^2}$ and $\beta = \arctan(b/a)$ (consider why?).

To rotate (a, b) in the counter-clockwise direction by degree θ is the same as multiplying $Re^{i\beta}$ by $e^{i\theta}$. Indeed,

$$Re^{i\beta}e^{i\theta} = Re^{i(\beta+\theta)}.$$

This is a vector with length R; and the angle between $Re^{i(\beta+\theta)}$ and the $x-$axis is $\beta + \theta$. Now let us compute this number

$$Re^{i\beta}e^{i\theta} = (a + ib)(\cos\theta + i\sin\theta) = a\cos\theta - b\sin\theta + i(a\sin\theta + b\cos\theta).$$

So rotating (a, b) in the counter-clockwise direction by degree θ, the resulting vector is

$$(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta),$$

or equivalently, in matrix language,

$$\begin{bmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Now we have proved that to rotate a vector (or equivalently a 2-column matrix) in $\mathbb{R}^2$ in the counter-clockwise direction by degree θ is the same as multiplying this column matrix.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

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